Mixed Fractional Integration and Differentiation as Reciprocal Operations

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Abstract: *We study the question of the composition of the mixed fractional integral and the mixed fractional derivative in a sufficiently broad class of functions. The treatment formula for the mixed fractional derivative is obtained.*
Keywords: *mixed fractional integral, mixed fractional derivative, the function of two variables, Riemann-Liouville integrals.*

1. Introduction

Various forms of fractional integrals and derivatives are known. Fractional integrals and Riemann-Liouville derivatives are the most common in the scientific literature [1]. Operators of generalized fractional integro-differentiation with Gauss hypergeometric function.
Direct extension of the Riemann-Liouville fractional integro-differentiation operations to the case of many variables, when these operators are applied for each variable or some of them, gives the so-called partial and mixed fractional integrals and derivatives. They are known [1], as well as [4], [5], [6], [7], [8], [9], [10], [11], [12], [13]. Thus, in [2], using the two-dimensional Laplace transform, a solution of the two-dimensional Abel integral equation was obtained.
In this paper, we study the question of the composition of the mixed fractional integral and the mixed fractional derivative in a sufficiently broad class of functions. The treatment formula for the mixed fractional derivative is obtained. The results obtained can be applied in the theory of differential equations containing the mixed fractional derivatives.
Lemma 3 on the representability of $f(x,y)\in AC^{n,m}(\overline{\Omega})$ function in the form of (6) and Lemma 4 generalized is the previously known Lemmas 1 and 2 for the two-dimensional case. Lemmas 3, 4 permits to prove the theorem (a necessary and sufficient condition for the representability of $f(x,y)$ function as the mixed fractional integral of a summable function) and Theorems 2 and 3 about the composition of a mixed fractional integral and a mixed fractional derivative. Note that Theorems 2 and 3 generalize the results of Theorem 2.4 [1, p. 44] for the two-dimensional case.

2. Preliminaries

The important role in the theory of fractional integro differentiation is played by absolutely continuous functions.
Let $\Omega=\{(x,y): a<x<b,\ c<y<d\}$,

$$-\infty\le a<b\le+\infty,\ -\infty\le c<d\le+\infty$$

Definition 1 [1, p. 2]. $f(x)$ the function is called absolutely non-discontinuous into segments $[a,b]$ if, for any, $\varepsilon>0$ there exists $\delta>0$ such that for any finite set of pairwise non-intersecting intervals $[a_k,b_k]\in[a,b]$, $k=\overline{1,m}$ such that $\sum_{k=1}^{m}(b_k-a_k)<\delta$ the inequality $\sum_{k=1}^{m}|f(b_k)-f(a_k)|<\varepsilon$ holds. The space of these functions is denoted by $AC([a,b])$.

Definition 2 [1, p. 2]. Let us denote by $AC^n([a,b])$, where $n=1,2,\ldots$, the spaces of functions $f(x)$ that have continuous derivatives up to order $n-1$ on $[a,b]$ with $f^{(n-1)}(x)\in AC([a,b])$.

Definition 3. A function $f(x,y)$ is called absolutely continuous in Ω if for any $\varepsilon>0$ there exists $\delta>0$ such that for any finite set of pairwise non-intersecting intervals $\Delta_k=\{(x,y): x_{1k}\le x\le x_{2k},\ y_{1k}\le y\le y_{2k}\}$, the sum of the areas of which is less δ, the inequality holds

$$\sum_{k=1}^{n}|f(x_{2k},y_{2k})-f(x_{2k},y_{1k})-f(x_{1k},y_{2k})+f(x_{1k},y_{1k})|<\varepsilon, \quad (1)$$

and if, moreover, $f(a,y)\in AC([c,d])$ and $f(x,c)\in AC([a,b])$. The class of all such functions is indicated $AC(\overline{\Omega})$.

Definition 4. By $AC^{n,m}(\overline{\Omega})$, where $n=1,2,\ldots$, let us denote the class of functions continuously differentiable on $\overline{\Omega}$ up to order $(n-1,\ m-1)$, and its mixed partial derivative $\frac{\partial^{n+m-2}f}{\partial x^{n-1}\partial y^{m-1}}$ is absolutely continuous in $\overline{\Omega}$.

It is known that the class $AC^n([a,b])$ belongs to those and only those functions $f(x)$ that are representable as antiderivatives of Lebesgue summable functions:

$$f(x)=\int_a^x \psi(x)dx+C, \quad \psi(x)\in L_1(a,b). \qquad (2)$$

Lemma 1 [1, p. 39]. Space $AC^n([a,b])$ consists of those and only those functions $f(x)$, which are represented in the form

$$f(x)=\frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}\varphi(t)dt+\sum_{k=0}^{n-1}C_k(x-a)^k, \qquad (3)$$

where $\varphi(x)\in L_1([a,b])$ C_k being arbitrary constants.

In the formula (3)

$$\varphi(t)=f^{(n)}(t), \quad C_k=\frac{f^{(k)}(a)}{k!}. \qquad (4)$$

The last equality uses the notation $f^{(n)}(x)=\dfrac{d^n f(x)}{dx^n}$.

A similar property of the functions $f(x,y)\in AC(\overline{\Omega})$ is as follows.

Lemma 2 [3, p. 238]. The class $AC(\overline{\Omega})$ consists of those and only those functions $f(x,y)$ which are represented in the form

$$f(x,y)=\int_a^x\int_c^y \varphi(t,s)dtds+\int_a^x \psi(t)dt+\int_c^y \eta(s)ds+C, \qquad (5)$$

where $\varphi(x,y)\in L_1(\Omega)$, $\psi(x)\in L_1([a,b])$, $\eta(y)\in L_1([c,d])$, and C is an arbitrary constant.

In order to generalize the last lemma to the case of a class $AC^{n,n}(\overline{\Omega})$, we need the following lemma.

Lemma 3. Let $f(x,y)\in AC(\overline{\Omega})$, then

$$f(x,y)=\frac{1}{(n-1)!(m-1)!}\int_a^x\int_c^y \frac{f^{(n,m)}(t,s)dtds}{(x-t)^{1-n}(y-s)^{1-m}}+$$
$$+\sum_{i=0}^{n-1}\frac{f^{(i,0)}(a,y)}{i!}(x-a)^i+\sum_{k=0}^{m-1}\frac{f^{(0,k)}(x,c)}{k!}(y-c)^k-$$
$$-\sum_{i=0}^{n-1}\sum_{k=0}^{m-1}\frac{f^{(i,k)}(a,c)}{i!k!}(x-a)^i(y-c)^k. \qquad (6)$$

In formula (6) the notation used

$$f^{(i,k)}(x,y)=\frac{d^{i+k}f(x,y)}{dx^i dy^k}.$$

Proof. Let be $\dfrac{\partial^{n+m-2}f}{\partial x^{n-1}\partial y^{m-1}}\in AC(\overline{\Omega})$. By virtue of Lemma 2, we have

$$\frac{\partial^{n+m-2}f}{\partial x^{n-1}\partial y^{m-1}}=\int_a^x\int_c^y \varphi(t,s)dtds+\int_a^x \psi(t)dt+\int_c^y \eta(s)ds+C_0 \qquad (7)$$

Integrating (7) times sequentially $n-1$ by x and times $m-1$ by y, we get

$$f(x,y)=\frac{1}{(n-1)!(m-1)!}\int_a^x\int_c^y (x-t)^{n-1}(y-s)^{m-1}\varphi(t,s)dtds+$$
$$+\frac{(y-c)^{m-1}}{(n-1)!(m-1)!}\int_a^x (x-t)^{n-1}\psi(t)dt+$$
$$+\frac{(x-a)^{n-1}}{(n-1)!(m-1)!}\int_c^y (y-s)^{m-1}\eta(s)ds+$$
$$+\sum_{i=0}^{n-1}\bar{\tau}_i(y)(x-a)^i+\sum_{k=0}^{m-1}\tilde{\tau}_k(x)(y-c)^k, \qquad (8)$$

where $\bar{\tau}_i(y)$ $(i=\overline{0,n-1})$, $\tilde{\tau}_k(x)$ $(k=\overline{0,m-1})$ is an arbitrary function. When integrating, the well-known for n - multiple integral formulae is used [1]

$$\int_a^x dx\int_a^x dx...\int_a^x F(x)dx=\frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}F(t)dt, \qquad (9)$$

proof, which is easy to implement by mathematical induction. It will be clear from the proof that an arbitrary constant in formula (7) is associated with arbitrary functions of formula (8) by the relation $(n-1)!\bar{\tau}_{n-1}^{(m-1)}(c)+(m-1)!\tilde{\tau}_{m-1}^{(n-1)}(a)=C_0$.

Since $f(x,y)\in AC^{n,m}(\overline{\Omega})$, then derivatives $\dfrac{\partial^{i+k}f}{\partial x^i\partial y^k}$ $(0\le i<n,\ 0\le k<m)$ exist and are continuous in Ω. Calculating the derivatives with x respect to the order $\overline{0,n-1}$ of the function $f(x,y)$ given by formula (8), and assuming in them $x=a$, we obtain the equalities

$$\frac{\partial^i f(a,y)}{\partial x^i}=i!\bar{\tau}_i(y)+\sum_{k=0}^{m-1}\tilde{\tau}_k^{(i)}(a)(y-c)^k, \quad i=\overline{0,n-2}, \qquad (10)$$

$$\frac{\partial^{n-1}f(a,y)}{\partial x^{n-1}}=\frac{1}{(m-1)!}\int_c^y \frac{\eta(s)ds}{(y-s)^{1-m}}+(n-1)!\bar{\tau}_{n-1}(y)+\sum_{k=0}^{m-1}\tilde{\tau}_k^{(n-1)}(a)(y-c)^k. \qquad (11)$$

Similarly, differentiating (8) by y and assuming $y=c$, we obtain the equality

$$\frac{\partial^k f(x,c)}{\partial y^k}=k!\tilde{\tau}_k(x)+\sum_{i=0}^{n-1}\bar{\tau}_i^{(k)}(c)(x-a)^i, \quad k=\overline{0,m-2}, \qquad (12)$$

$$\frac{\partial^{m-1}f(x,c)}{\partial y^{m-1}}=\frac{1}{(n-1)!}\int_a^x \frac{\psi(t)dt}{(x-t)^{1-n}}+(m-1)!\tilde{\tau}_{m-1}(y)+\sum_{i=0}^{n-1}\bar{\tau}_i^{(m-1)}(c)(x-a)^i. \qquad (13)$$

Expressing from formulas (10) - (13) $\bar{\tau}_i(y)$ and $\tilde{\tau}_k(x)$ respectively, we get

$$\sum_{i=0}^{n-1}\bar{\tau}_i(y)(x-a)^i+\sum_{k=0}^{m-1}\tilde{\tau}_k(x)(y-c)^k=$$

$$=\sum_{i=0}^{n-1}\frac{(x-a)^i}{i!}\left(\frac{\partial^i f(a,y)}{\partial x^i}-\sum_{k=0}^{m-1}\tilde{\tau}_k^{(i)}(x)(y-c)^k\right)+$$

$$+\sum_{k=0}^{m-1}\frac{(y-c)^k}{k!}\left(\frac{\partial^k f(x,c)}{\partial y^k}-\sum_{i=0}^{n-1}\bar{\tau}_i^{(k)}(y)(x-a)^i\right)-$$

$$-\frac{(x-a)^{n-1}}{(n-1)!(m-1)!}\int_c^y\frac{\eta(s)ds}{(y-s)^{1-m}}-$$

$$-\frac{(y-c)^{m-1}}{(n-1)!(m-1)!}\int_a^x\frac{\psi(t)dt}{(x-t)^{1-n}}=\sum_{i=0}^{n-1}\frac{(x-a)^i}{i!}\frac{\partial^i f(a,y)}{\partial x^i}+$$

$$+\sum_{k=0}^{m-1}\frac{(y-c)^k}{k!}\frac{\partial^k f(x,c)}{\partial y^k}-$$

$$-\frac{(x-a)^{n-1}}{(n-1)!(m-1)!}\int_c^y\frac{\eta(s)ds}{(y-s)^{1-m}}-\frac{(y-c)^{m-1}}{(n-1)!(m-1)!}\int_a^x\frac{\psi(t)dt}{(x-t)^{1-n}}-$$

$$-\sum_{i=0}^{n-1}\sum_{k=0}^{m-1}(x-a)^i(y-c)^k\left(\frac{\bar{\tau}_i^{(k)}(c)}{k!}+\frac{\tilde{\tau}_k^{(i)}(a)}{i!}\right).\qquad(14)$$

Calculating the mixed derivatives $\frac{\partial^{i+k}f}{\partial x^i\partial y^k}$ of the function (8) at a point (a,c), we get

$$\frac{1}{i!k!}\frac{\partial^{i+k}f(a,c)}{\partial x^i\partial y^k}=\frac{\bar{\tau}_i^{(k)}(c)}{k!}+\frac{\tilde{\tau}_k^{(i)}(a)}{i!}.\qquad(15)$$

Substituting (14), (15) into (8), we get

$$f(x,y)=\frac{1}{(n-1)!(m-1)!}\int_a^x\int_c^y\frac{\varphi(t,s)dtds}{(x-t)^{1-n}(y-s)^{1-m}}+$$

$$+\sum_{i=0}^{n-1}\frac{1}{i!}\frac{\partial^i f(a,y)}{\partial x^i}(x-a)^i+\sum_{k=0}^{m-1}\frac{1}{k!}\frac{\partial^k f(x,c)}{\partial y^k}(y-c)^k-$$

$$-\sum_{i=0}^{n-1}\sum_{k=0}^{m-1}\frac{1}{i!k!}\frac{\partial^{i+k}f(a,c)}{\partial x^i\partial y^k}(x-a)^i(y-c)^k.\quad(16)$$

Equality (6) follows from (16) and from the fact that $\varphi(x,y)=\frac{\partial^{m+n}f(a,c)}{\partial x^n\partial y^m}$. The lemma is proved.

The following lemma gives a description of the class $AC^{n,m}(\bar{\Omega})$. It generalizes Lemma 1 to the case of two variables and Lemma 2 to the case $n+m>2$.

Lemma 4. Space $AC^{n,m}(\bar{\Omega})$ consists of those and only those functions $f(x,y)$, which are represented in the form

$$f(x,y)=\frac{1}{(n-1)!(m-1)!}\int_a^x\int_c^y(x-t)^{n-1}(y-s)^{m-1}\varphi(t,s)dtds+$$

$$+\sum_{k=0}^{m-1}\frac{(y-c)^k}{(n-1)!k!}\int_a^x(x-t)^{n-1}\psi_k(t)dt+$$

$$+\sum_{i=0}^{n-1}\frac{(x-a)^i}{i!(m-1)!}\int_c^y(y-s)^{m-1}\eta(s)ds+$$

$$+\sum_{i=0}^{n-1}\sum_{k=0}^{m-1}C_{ik}(x-a)^i(y-c)^k,\qquad(17)$$

where $\varphi(x,y)\in L_1(\Omega)$, $\psi_k(x)\in L_1([a,b])$ $\left(k=\overline{0,m-1}\right)$, $\eta(y)\in L_1([c,d])$, $\left(i=\overline{0,n-1}\right)$, C_{ik} being arbitrary constants.

Proof. *Necessity*. Let $f(x,y)\in AC^{n,m}(\bar{\Omega})$. According to the lemma 3

$$f(x,y)=\frac{1}{(n-1)!(m-1)!}\int_a^x\int_c^y\frac{f^{(n,m)}(t,s)}{(x-t)^{1-n}(y-s)^{1-m}}dtds+$$

$$+\sum_{i=0}^{n-1}\frac{f^{(i,0)}(a,y)}{i!}(x-a)^i+\sum_{k=0}^{m-1}\frac{f^{(0,k)}(x,c)^i}{k!}(y-c)^k-$$

$$-\sum_{i=0}^{n-1}\sum_{k=0}^{m-1}\frac{f^{(i,k)}(a,c)}{i!k!}(x-a)^i(y-c)^k.\qquad(18)$$

Because $f^{(n-1,m-1)}(x,y)\in AC(\bar{\Omega})$, then $f^{(n-1,m-1)}(a,y)\in AC([c,d])$, consequently, $f^{(n-1,0)}(a,y)\in AC^m([c,d])$, from here $f^{(i,0)}(a,y)\in AC^m([c,d])$ $\left(i=\overline{0,n-1}\right)$. Use lemma [1, c.39]

$$f^{(i,0)}(a,y)=\frac{1}{(m-1)!}\int_c^y\frac{\eta_i(s)}{(y-s)^{1-m}}ds+\sum_{k=0}^{m-1}\frac{f^{(i,k)}(a,c)}{k!}(y-c)^k\qquad(19)$$

where $\eta_i(y)\in L_1([c,d])$. Then

$$\sum_{i=0}^{n-1}\frac{f^{(i,0)}(a,y)}{i!}(x-a)^i=\sum_{i=0}^{n-1}\frac{(x-a)^i}{i!\,(m-1)!}\int_c^y\frac{\eta_i(s)}{(y-s)^{1-m}}ds+$$

$$+\sum_{i=0}^{n-1}\sum_{k=0}^{m-1}\frac{f^{(i,k)}(a,c)}{i!\,k!}(x-a)^i(y-c)^k.$$

(20)

Similarly, it is proved that

$$\sum_{k=0}^{m-1}\frac{f^{(0,k)}(x,c)}{k!}(y-c)^k=\sum_{k=0}^{m-1}\frac{(y-c)^k}{ik!\,(n-1)!}\int_a^x\frac{\psi_k(t)}{(x-t)^{1-m}}dt+$$

$$+\sum_{i=0}^{n-1}\sum_{k=0}^{m-1}\frac{f^{(i,k)}(a,c)}{i!\,k!}(x-a)^i(y-c)^k.$$

(21)

where $\psi_k(x)\in L_1([a,b])$. Substituting (20), (21) into (18), we obtain the formula (17), in which

$$C_{ik}=\frac{1}{i!k!}f^{(i,k)}(a,c).\qquad(22)$$

Sufficiency. When calculating directly $\frac{\partial^{i+k}f}{\partial x^i\partial y^k}$ $(0\le i<n,0\le k<m)$, it is easy to make sure that they are all continuous in $\bar{\Omega}$, and

$$\frac{\partial^{n+m-2}f}{\partial x^{n-1}\partial y^{m-1}}=\int_a^x\int_c^y\varphi(t,s)dtds+\int_a^x\psi(t)dt+\int_c^y\eta(s)ds+\qquad(23)$$
$$+(n-1)!\,(m-1)!c_{n-1,m-1}.$$

Obviously $\frac{\partial^{n+m-2} f}{\partial x^{n-1}\partial y^{m-1}} \in AC(\overline{\Omega})$, from where it follows $f(x,y)\in AC^{n,m}(\overline{\Omega})$.

The theorem is proven completely.

Notice, that

$$\varphi(x,y)= f^{(n,m)}(x,y); \quad (24)$$

$$\psi_k(x)= f^{(n,k)}(x,c), \quad k=\overline{0,m-1}; \quad (25)$$

$$\eta_i(y)= f^{(i,m)}(a,y), \quad i=\overline{0,n-1}; \quad (26)$$

$$C_{ik}=\frac{1}{i!k!} f^{(i,k)}(a,c). \quad (27)$$

Definition 5 [1, c. 459]. Let $f(x,y)\in L_1(\Omega)$. The integral

$$\left(I^{\alpha,\beta}_{a+,c+} f\right)(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_c^y \frac{f(t,s)dtds}{(x-t)^{1-\alpha}(y-s)^{1-\beta}}, \quad (28)$$

where $\alpha>0, \beta>0$, is called a left-hand sided mixed Riemann-Liouville fractional integral of order (α,β).

The fractional integral (28) is obviously defined on functions $f(x,y)\in L_1(\Omega)$, existing almost everywhere.

Using the Fubini theorem, the semigroup property is proved.

Let $f(x,y)\in L_1(\Omega)$, $\alpha,\beta,\gamma,\delta$ be positive numbers, then equality holds almost everywhere in Ω

$$I^{\alpha,\beta}_{a+,c+} I^{\gamma,\delta}_{a+,c+} f = I^{\alpha+\gamma,\beta+\delta}_{a+,c+} f. \quad (29)$$

It can be shown that if $\alpha>0$ the function $f(x,y)$ is defined in Ω and $f(x,y)\in L_1(\Omega)$, then

$$\left(I^{\alpha}_{a+,x}\right)(x,y)\in L_1([c,d]) \quad \forall x\in(a,b);$$

$$\left(I^{\alpha}_{a+,y}\right)(x,y)\in L_1([a,b]) \quad \forall y\in(c,d).$$

In the last equations $I^{\alpha}_{a+,x} f$, $I^{\alpha}_{a+,y} f$ are partial Riemann – Liouville fractional integrals with respect to the variables x and y, respectively.

Taking these equalities into account, it is directly verified that

$$\left(I^{\alpha}_{a+,x} I^{\beta}_{c+,y} f\right)(x,y)=\left(I^{\beta}_{c+,y} I^{\alpha}_{a+,x} f\right)(x,y)=\left(I^{\alpha,\beta}_{a+,c+} f\right)(x,y). \quad (30)$$

Definition 6 [1, c. 460]. For function $f(x,y)$, given on Ω, formula

$$\left(D^{\alpha,\beta}_{a+,c+} f\right)(x,y)=\frac{1}{\Gamma(n-\alpha)\Gamma(m-\beta)}\times \frac{\partial^{n+m}}{\partial x^n \partial y^m}\int_a^x\int_c^y \frac{f(t,s)dtds}{(x-t)^{\alpha-n+1}(y-s)^{\beta-m+1}} \quad (31)$$

where $\alpha>0, \beta>0$, is called a mixed Riemann-Liouville fractional derivative of order (α,β), $n=[\alpha]+1$, $m=[\beta]+1$.

If the function $f(x,y)$ has a property $I^{n-\alpha,m-\beta}_{a+,c+} f\in AC^{n,m}(\overline{\Omega})$, then the order of taking the derivatives in (31) does not matter, and $\left(D^{\alpha,\beta}_{a+,c+} f\right)(x,y)\in L_1(\Omega)$.

Definition 7 is a two-dimensional analogue of Definition 2.3 [1, p. 43].

3. Compositions of mixed fractional integral and mixed fractional derivative of the same order

Following [1, p. 44], we define the following classes of functions.

Definition 7. Let $I^{\alpha,\beta}_{a+,c+}(L_1)$ denote the space of function $f(x,y)$, represented by the left-sided mixed fractional integral of the order (α,β) of a summable function: $f=I^{\alpha,\beta}_{a+,c+}\varphi$, $\varphi\in L_1(\Omega)$.

Definition 8. Let $0<\alpha<1, 0<\beta<1$. A function $f(x,y)\in L_1(\Omega)$ is said to have a summable fractional derivative $D^{\alpha,\beta}_{a+,c+} f$, if $I^{n-\alpha,m-\beta}_{a+,c+} f\in AC^{n,m}(\overline{\Omega})$.

The following theorem defines the necessary and sufficient condition for the unique solvability of the two-dimensional Abel integral equation.

Theorem 1. In order that $f(x,y)\in I^{\alpha,\beta}_{a+,c+}(L_1)$, $\alpha>0, \beta>0$, it is necessary and sufficient that

$$f_{n-\alpha,m-\beta}\in AC^{n,m}(\overline{\Omega}), \quad (32)$$

where $n=[\alpha]+1$, $m=[\beta]+1$, and that

$$f^{(i,0)}_{n-\alpha,m-\beta}(a,y)\equiv 0, \quad i=\overline{0,n-1}; \quad (33)$$

$$f^{(0,k)}_{n-\alpha,m-\beta}(x,c)\equiv 0, \quad k=\overline{0,m-1}; \quad (34)$$

$$f^{(i,k)}_{n-\alpha,m-\beta}(a,c)\equiv 0, \quad i=\overline{0,n-1}, \ k=\overline{0,m-1}. \quad (35)$$

Proof. *Necessity.* Let $f=I^{\alpha,\beta}_{a+,c+}\varphi$, $\varphi\in L_1(\Omega)$. In view of the semigroup property

$$f_{n-\alpha,m-\beta}(x,y)=I^{n-\alpha,m-\beta}_{a+,c+} f = I^{\alpha,\beta}_{a+,c+}\varphi, \quad (36)$$

where $\varphi\in L_1(\Omega)$. From here follow feasibility conditions (33) – (35). Feasibility condition (32) follow from Lemma 4.

This implies the fulfillment of conditions (33) - (35). The fulfillment of condition (32) follows from Lemma 4.

Sufficiency. Under condition (32), we can present $f_{n-\alpha,m-\beta}$ according to Lemma 3, in the form

$$f_{n-\alpha,m-\beta}(x,y)=\frac{1}{(n-1)!(m-1)!}\int_a^x\int_c^y \frac{f^{(n,m)}_{n-\alpha,\, m-\beta}(t,s)}{(x-t)^{1-n}(y-s)^{1-m}}dtds+$$
$$+\sum_{i=0}^{n-1}\frac{f^{(i,0)}_{n-\alpha,m-\beta}(a,y)}{i!}(x-a)^i+\sum_{k=0}^{m-1}\frac{f^{(0,k)}_{n-\alpha,m-\beta}(x,c)}{k!}(y-c)^k-$$
$$-\sum_{i=0}^{n-1}\sum_{k=0}^{m-1}\frac{f^{(i,k)}_{n-\alpha,m-\beta}(a,c)}{k!}(x-a)^i(y-c)^k, \quad (37)$$

where $f^{(n,m)}_{n-\alpha,m-\beta}\in L_1(\Omega)$. Taking into account conditions (33) - (35), the last equality is written in the form

$$f_{n-\alpha,m-\beta}(x,y)=\frac{1}{(n-1)!(m-1)!}\int_a^x\int_c^y\frac{f^{(n,m)}_{n-\alpha,m-\beta}(t,s)}{(x-t)^{1-n}(y-s)^{1-m}}dtds$$
(38)

Using the semigroup property (29), we can write
$$I^{n-\alpha,m-\beta}_{a+,c+}f=I^{n,m}_{a+,c+}f^{(n,m)}_{n-\alpha,m-\beta}=I^{n-\alpha,m-\beta}_{a+,c+}I^{\alpha,\beta}_{a+,c+}f^{(n,m)}_{n-\alpha,m-\beta}.$$
(39)

From here $I^{n-\alpha,m-\beta}_{a+,c+}\left(f-I^{\alpha,\beta}_{a+,c+}f^{(n,m)}_{n-\alpha,m-\beta}\right)=0$. Applying the integral to this equality $I^{\alpha,\beta}_{a+,c+}$, we get
$$I^{n,m}_{a+,c+}\left(f-I^{\alpha,\beta}_{a+,c+}f^{(n,m)}_{n-\alpha,m-\beta}\right)dxdy=0. \qquad (40)$$
From here $f=I^{\alpha,\beta}_{a+,c+}f^{(n,m)}_{n-\alpha,m-\beta}$, $f^{(n,m)}_{n-\alpha,m-\beta}\in L_1(\Omega)$. The theorem is proved.

Note that Theorem 1 is a generalization of Theorem 2.3 [1, p. 43] in the case of two variables. From it, in particular, it follows that the class of functions having a summable fractional derivative $\mathrm{D}^{\alpha,\beta}_{a+,c+}f$ in the sense of Definition 8 is wider than the class of functions $I^{\alpha,\beta}_{a+,c+}(L_1)$. Namely, the class $I^{\alpha,\beta}_{a+,c+}(L_1)$ owns only those functions that have a summable fractional derivative $\mathrm{D}^{\alpha,\beta}_{a+,c+}f$, for which equalities (33) - (35) hold.

Theorem 2. Let $\alpha>0,\beta>0$. Then equality
$$\mathrm{D}^{\alpha,\beta}_{a+,c+}I^{\alpha,\beta}_{a+,c+}f=f(x,y) \qquad (41)$$
performed for any summable function $f(x,y)$.

Proof. We have
$$\mathrm{D}^{\alpha,\beta}_{a+,c+}I^{\alpha,\beta}_{a+,c+}f=\frac{\partial^{n+m}}{\partial x^n\partial y^m}I^{n-\alpha,m-\beta}_{a+,c+}I^{\alpha,\beta}_{a+,c+}f=$$
$$=\frac{1}{\Gamma(\alpha)\Gamma(\beta)\Gamma(n-\alpha)\Gamma(m-\beta)}\frac{\partial^{n+m}}{\partial x^n\partial y^m}\int_a^x\int_c^y\frac{dtds}{(x-t)^{\alpha}(y-s)^{\beta}}\times$$
$$\times\int_a^t\int_c^s\frac{f(u,v)dudv}{(t-u)^{n-\alpha}(s-v)^{m-\beta}} \qquad (42)$$
Changing the order of integration, we get
$$\mathrm{D}^{\alpha,\beta}_{a+,c+}I^{\alpha,\beta}_{a+,c+}f=\frac{(\Gamma(\alpha)\Gamma(\beta))^{-1}}{\Gamma(n-\alpha)\Gamma(m-\beta)}\frac{\partial^{n+m}}{\partial x^n\partial y^m}\int_a^x\int_c^y f(u,v)dudv\times$$
$$\times\int_u^x\int_v^y\frac{dtds}{(x-t)^{\alpha}(y-s)^{\beta}(t-u)^{n-\alpha}(s-v)^{m-\beta}}=$$
$$=\frac{\partial^{n+m}}{\partial x^n\partial y^m}\int_a^x\int_c^y f(u,v)dudv\frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)}\times$$
$$\times\int_u^x\frac{dt}{(x-t)^{\alpha}(t-u)^{n-\alpha}}\frac{1}{\Gamma(\beta)\Gamma(m-\beta)}\int_v^y\frac{ds}{(y-s)^{\beta}(s-v)^{m-\beta}}=$$
$$=\frac{1}{\Gamma(n)\Gamma(m)}\frac{\partial^{n+m}}{\partial x^n\partial y^m}\int_a^x\int_c^y\frac{f(u,v)}{(x-u)^{1-n}(y-v)^{1-m}}dudv=$$
$$=f(x,y), \qquad (43)$$
Q.E.D.

Theorem 3. For any function $f(x,y)\in I^{\alpha,\beta}_{a+,c+}(L_1)$ the equality
$$I^{\alpha,\beta}_{a+,c+}\mathrm{D}^{\alpha,\beta}_{a+,c+}f=f(x,y), \qquad (44)$$
and for any function that has a summable derivative $\mathrm{D}^{\alpha,\beta}_{a+,c+}f$ (in the sense of definition 8), the equality
$$I^{\alpha,\beta}_{a+,c+}\mathrm{D}^{\alpha,\beta}_{a+,c+}f=f(x,y)-\sum_{i=0}^{n-1}\frac{(x-a)^{\alpha-i-1}}{\Gamma(\alpha-i)}f^{(n-i-1,0)}_{n-\alpha,0}(a,y)-$$
$$-\sum_{k=0}^{m-1}\frac{(y-c)^{\beta-k-1}}{\Gamma(\beta-k)}f^{(0,m-k-1)}_{0,m-\beta}(x,c)+$$
$$+\sum_{i=0}^{n-1}\sum_{k=0}^{m-1}\frac{(x-a)^{\alpha-i-1}(y-c)^{\beta-k-1}}{\Gamma(\alpha-i)\Gamma(\beta-k)}f^{(n-i-1,m-k-1)}_{n-\alpha,m-\beta}(a,c),$$
(45)

where $f_{\gamma,\delta}(x,y)=I^{\gamma,\delta}_{a+,c+}f$.

Proof. Let $f(x,y)\in I^{\alpha,\beta}_{a+,c+}(L_1)$, then $f(x,y)=I^{\alpha,\beta}_{a+,c+}\varphi$, $\varphi(x,y)\in L_1(\Omega)$. Based on Theorem 2, we have
$$I^{\alpha,\beta}_{a+,c+}\mathrm{D}^{\alpha,\beta}_{a+,c+}f=I^{\alpha,\beta}_{a+,c+}\mathrm{D}^{\alpha,\beta}_{a+,c+}I^{\alpha,\beta}_{a+,c+}\varphi=I^{\alpha,\beta}_{a+,c+}\varphi=f(x,y).$$
(46)

Let now $I^{1-\alpha,1-\beta}_{a+,c+}f\in AC(\overline{\Omega})$. According to Lemma 3, the integral $f_{n-\alpha,m-\beta}(x,y)=I^{n-\alpha,m-\beta}_{a+,c+}f$ can be represented as
$$f_{n-\alpha,m-\beta}(x,y)=I^{n,m}_{a+,c+}f^{(n,m)}_{n-\alpha,m-\beta}+\sum_{i=0}^{n-1}\frac{f^{(i,0)}_{n-\alpha,m-\beta}(a,y)}{i!}(x-a)^i+$$
$$+\sum_{k=0}^{m-1}\frac{f^{(0,k)}_{n-\alpha,m-\beta}(x,c)}{k!}(y-c)^k-$$
$$-\sum_{i=0}^{n-1}\sum_{k=0}^{m-1}\frac{f^{(i,k)}_{n-\alpha,m-\beta}(a,c)}{k!i!}(x-a)^i(y-c)^k. \qquad (47)$$
By the semigroup property, the equality
$$I^{n,m}_{a+,c+}f^{(n,m)}_{n-\alpha,m-\beta}=I^{n-\alpha,m-\beta}_{a+,c+}I^{\alpha,\beta}_{a+,c+}f^{(n,m)}_{n-\alpha,m-\beta}. \qquad (48)$$
Further,
$$\frac{(x-a)^i}{i!}f^{(i,0)}_{n-\alpha,m-\beta}(a,y)=$$
$$=I^{n-\alpha,m-\beta}_{a+,c+}\left(\mathrm{D}^{n-\alpha}_{a+,x}\frac{(x-a)^i}{i!}\mathrm{D}^{m-\beta}_{c+,y}f^{(i,0)}_{n-\alpha,m-\beta}(a,y)\right)+$$
$$+\frac{(x-a)^i(y-c)^{m-\beta-1}}{i!\Gamma(m-\beta)}f^{(i,0)}_{n-\alpha,1}(a,c)=$$
$$=I^{n-\alpha,m-\beta}_{a+,c+}\left(\frac{(x-a)^{i-n+\alpha}}{\Gamma(1+i-n+\alpha)}f^{(i,0)}_{n-\alpha,0}(a,y)\right)+$$
$$+\frac{(x-a)^i(y-c)^{m-\beta-1}}{i!\Gamma(m-\beta)}f^{(i,0)}_{n-\alpha,1}(a,c). \qquad (49)$$
From the last equality it follows that
$$\sum_{i=0}^{n-1}\frac{f^{(i,0)}_{n-\alpha,m-\beta}(a,y)}{i!}(x-a)^i=$$
$$=I^{n-\alpha,m-\beta}_{a+,c+}\left(\sum_{i=0}^{n-1}\frac{(x-a)^{i-n+\alpha}}{\Gamma(1+i-n+\alpha)}f^{(i,0)}_{n-\alpha,0}(a,y)\right)+$$

$$+\sum_{i=0}^{n-1}\frac{(x-a)^i(y-c)^{m-\beta-1}}{i!\Gamma(m-\beta)}f_{n-\alpha,1}^{(i,0)}(a,c), \qquad (50)$$

from where, redesignating the summation index, we get

$$\sum_{i=0}^{n-1}\frac{f_{n-\alpha,m-\beta}^{(i,0)}(a,y)}{i!}(x-a)^i=$$
$$=I_{a+,c+}^{n-\alpha,m-\beta}\left(\sum_{i=0}^{n-1}\frac{(x-a)^{\alpha-i-1}}{\Gamma(\alpha-i)}f_{n-\alpha,0}^{(n-i-1,0)}(a,y)\right)+$$
$$+\sum_{i=0}^{n-1}\frac{(x-a)^i(y-c)^{m-\beta-1}}{i!\Gamma(m-\beta)}f_{n-\alpha,1}^{(i,0)}(a,c). \qquad (51)$$

Equality is obtained similarly

$$\sum_{i=0}^{n-1}\frac{f_{n-\alpha,m-\beta}^{(0,k)}(x,c)}{i!}(y-c)^k=$$
$$=I_{a+,c+}^{n-\alpha,m-\beta}\left(\sum_{k=0}^{m-1}\frac{(y-c)^{\beta-k-1}}{\Gamma(\beta-k)}f_{0,m-\beta}^{(0,m-k-1)}(x,c)\right)+$$
$$+\sum_{k=0}^{m-1}\frac{(x-a)^{n-\alpha-1}(y-c)^k}{k!\Gamma(n-\alpha)}f_{1,m-\beta}^{(0,k)}(a,c). \qquad (52)$$

It is not difficult to see that

$$\sum_{i=0}^{n-1}\sum_{i=0}^{n-1}\frac{f_{n-\alpha,m-\beta}^{(i,k)}(a,c)}{i!k!}(x-a)^i(y-c)^k=$$
$$=I_{a+,c+}^{n-\alpha,m-\beta}\left(\sum_{i=0}^{n-1}\sum_{k=0}^{m-1}I_{a+,c+}^{n-\alpha,m-\beta}\left(\frac{(x-a)^i(y-c)^k}{i!\,k!}\right)f_{n-\alpha,m-\beta}^{(i,k)}(a,c)\right)=$$
$$=I_{a+,c+}^{n-\alpha,m-\beta}\left(\sum_{i=0}^{n-1}\sum_{k=0}^{m-1}\frac{(x-a)^{\alpha-i-1}(y-c)^{\beta-k-1}}{\Gamma(\alpha-i)\Gamma(\beta-k)}f_{n-\alpha,m-\beta}^{(n-i-1,m-k-1)}(a,c)\right).$$

(53)

Taking into account equalities (48), (51) - (53), equality (47) is written in the form

$$I_{a+,c+}^{n-\alpha,m-\beta}f=I_{a+,c+}^{n-\alpha,m-\beta}I_{a+,c+}^{\alpha,\beta}\mathrm{D}_{a+,c+}^{\alpha,\beta}f+$$
$$+I_{a+,c+}^{n-\alpha,m-\beta}\left(\sum_{i=0}^{n-1}\frac{(x-a)^{\alpha-i-1}}{\Gamma(\alpha-i)}f_{n-\alpha,0}^{(n-i-1,0)}(a,y)\right)+$$
$$+\sum_{i=0}^{n-1}\frac{(x-a)^i(y-c)^{m-\beta-1}}{i!\Gamma(m-\beta)}f_{n-\alpha,1}^{(i,0)}(a,c)+$$
$$+I_{a+,c+}^{n-\alpha,m-\beta}\left(\sum_{k=0}^{m-1}\frac{(y-c)^{\beta-k-1}}{\Gamma(\beta-k)}f_{0,m-\beta}^{(0,m-k-1)}(x,c)\right)+$$
$$+\sum_{k=0}^{m-1}\frac{(x-a)^{n-\alpha-1}(y-c)^k}{k!\Gamma(n-\alpha)}f_{1,m-\beta}^{(0,k)}(a,c)-$$
$$-I_{a+,c+}^{n-\alpha,m-\beta}\left(\sum_{i=0}^{n-1}\sum_{k=0}^{m-1}\frac{(x-a)^{\alpha-i-1}(y-c)^{\beta-k-1}}{\Gamma(\alpha-i)\Gamma(\beta-k)}f_{n-\alpha,m-\beta}^{(n-i-1,m-k-1)}(a,c)\right)$$

. (54)

By grouping the terms, we get

$$I_{a+,c+}^{n-\alpha,m-\beta}\left(f-I_{a+,c+}^{\alpha,\beta}\mathrm{D}_{a+,c+}^{\alpha,\beta}f-\sum_{i=0}^{n-1}\frac{(x-a)^{\alpha-i-1}}{\Gamma(\alpha-i)}f_{n-\alpha,0}^{(n-i-1,0)}(a,y)-\right.$$
$$-\sum_{k=0}^{m-1}\frac{(y-c)^{\beta-k-1}}{\Gamma(\beta-k)}f_{0,m-\beta}^{(0,m-k-1)}(x,c)+$$
$$\left.+\sum_{i=0}^{n-1}\sum_{k=0}^{m-1}\frac{(x-a)^{\alpha-i-1}(y-c)^{\beta-k-1}}{\Gamma(\alpha-i)\Gamma(\beta-k)}f_{n-\alpha,m-\beta}^{(n-i-1,m-k-1)}(a,c)\right)=$$
$$=\sum_{i=0}^{n-1}\frac{(x-a)^i(y-c)^{m-\beta-1}}{i!\,\Gamma(m-k)}f_{n-\alpha,1}^{(i,0)}(a,c)+$$
$$+\sum_{k=0}^{m-1}\frac{(y-c)^k(x-a)^{n-\alpha-1}}{k!\,\Gamma(n-\alpha)}f_{1,m-\beta}^{(0,k)}(a,c) \qquad (55)$$

In the right-hand side of equality (55), under the integral is a summable function. Applying the operator $I_{a+,c+}^{\alpha,\beta}$ to both parts of equality (55), we obtain

$$I_{a+,c+}^{n,m}\left(f-I_{a+,c+}^{\alpha,\beta}\mathrm{D}_{a+,c+}^{\alpha,\beta}f-\sum_{i=0}^{n-1}\frac{(x-a)^{\alpha-i-1}}{\Gamma(\alpha-i)}f_{n-\alpha,0}^{(n-i-1,0)}(a,y)-\right.$$
$$-\sum_{k=0}^{m-1}\frac{(y-c)^{\beta-k-1}}{\Gamma(\beta-k)}f_{0,m-\beta}^{(0,m-k-1)}(x,c)+$$
$$\left.+\sum_{i=0}^{n-1}\sum_{k=0}^{m-1}\frac{(x-a)^{\alpha-i-1}(y-c)^{\beta-k-1}}{\Gamma(\alpha-i)\Gamma(\beta-k)}f_{n-\alpha,m-\beta}^{(n-i-1,m-k-1)}(a,c)\right)=$$
$$=\sum_{i=0}^{n-1}\frac{(x-a)^{i+\alpha}(y-c)^{m-1}}{\Gamma(i+\alpha+1)\Gamma(m)}f_{n-\alpha,1}^{(i,0)}(a,c)+$$
$$+\sum_{k=0}^{m-1}\frac{(y-c)^{k+\beta}(x-a)^{n-1}}{\Gamma(k+1+\beta)\Gamma(n)}f_{1,m-\beta}^{(0,k)}(a,c) \quad . \qquad (56)$$

Under the integral on the left side of the equality is the summable function, and the right side of the equality is absolutely continuous. Finding the mixed derivative $\frac{\partial^{n+m}}{\partial x^n\partial y^m}$ of both parts of the equality, we get

$$f-I_{a+,c+}^{\alpha,\beta}\mathrm{D}_{a+,c+}^{\alpha,\beta}f-\sum_{i=0}^{n-1}\frac{(x-a)^{\alpha-i-1}}{\Gamma(\alpha-i)}f_{n-\alpha,0}^{(n-i-1,0)}(a,y)-$$
$$-\sum_{k=0}^{m-1}\frac{(y-c)^{\beta-k-1}}{\Gamma(\beta-k)}f_{0,m-\beta}^{(0,m-k-1)}(x,c)+$$
$$+\sum_{i=0}^{n-1}\sum_{k=0}^{m-1}\frac{(x-a)^{\alpha-i-1}(y-c)^{\beta-k-1}}{\Gamma(\alpha-i)\Gamma(\beta-k)}f_{n-\alpha,m-\beta}^{(n-i-1,m-k-1)}(a,c)=0\,.$$

(57)

The theorem is proved.

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